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M-estimator with asymmetric influence function for estimating the Burr type III parameters with outliers

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ABSTRACT

The Burr type III distribution allows for a wider region for the skewness and kurtosis plane, which covers several distributions including the log-logistic, and the Weibull and Burr type XII distributions. However, outliers may occur in the data set. The robust regression method such as an M-estimator with symmetric influence function has been successfully used to diminish the effect of outliers on statistical inference. However, when the data distribution is asymmetric, these methods yield biased estimators. We present an M-estimator with asymmetric influence function (AM-estimator) based on the quantile function of the Burr type III distribution to estimate the parameters for complete data with outliers. The simulation results show that the M-estimator with asymmetric influence function generally outperforms the maximum likelihood and traditional M-estimator methods in terms of the bias and root mean square errors. One real example is used to demonstrate the performance of our proposed method.

1. Introduction

Burr [1] provided several forms of cumulative distribution functions for fitting data. The Burr type III distribution, in particular, covers a wider region in the skewness and kurtosis plane including the gamma family, the Weibull family, the log-normal family, the bell-shaped and J-shaped beta distributions (see [2]). The Burr type III distribution also includes the entire region covered by the Burr XII distribution. Thus, the Burr type III distribution could be used as an alternative distribution for fitting data. Lindsay et al. [3] compared the Burr type III distribution with the Weibull distribution for the diameter data from 20 permanent sample plots of pinus radiate. The results showed that the Burr type III distribution outperformed the Weibull distribution in its ability to summarize diameter data. Clark et al. [4] used the maximum likelihood method to fit a sample of 1034 fault-trace lengths from the South Yorkshire coalfields. The results showed that the Burr type III distribution is capable of providing a satisfactory fit for the data. Shao [5] investigated the constrained maximum likelihood method to estimate the parameters of the Burr type III distribution for toxicity data. He also provided the lower confidence limits of percentile estimates using the delta method. Recently, Shao et al. [6] provided a modified Burr type III distribution for low-flow frequency data; it was a heavy lower tail distribution. A simulation study was conducted to investigate the performance of parameter estimates using three different methods. The results showed that the maximum likelihood estimates (MLE) method outperformed the other two methods—the method of moments and the probability-weighted moments. Gove et al. [7] reported that the Burr type III is suitable for fitting data from uneven-aged northern hardwood stands.

However, there is a possibility that the data collection contained outliers. The MLE method's parameter estimation is very sensitive to outliers. The robust regression (RR) method has been successful in tracing outliers, and making sure that

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they do not affect the statistical inference. RR methods that can be used to trace outliers include the likes of least absolute value, least trimmed squares, the M-estimator, and the GM-estimator. Huber [8–10] developed a series of estimations called M-estimations, which are based on a variety of objective functions. Traditional M-estimator methods based on symmetric robustifying functions assume that the distribution is symmetric, but when the data distribution is asymmetric, these methods yield biased estimators (see [11]). Bustos et al. [12] investigated the M-estimators for the parameters of the G_A^0 distribution. We present an M-estimator with asymmetric influence function (AM-estimator) based on the quantile function of the Burr type III distribution to estimate the parameters for complete data with outliers. We also compare our proposed AM-estimator with the MLE and traditional M-estimator methods. The confidence interval of the M-estimator with asymmetric influence function method is also provided. To demonstrate the performance of the proposed method, the Monte Carlo simulation study is presented and one numerical example is also discussed. Finally, we make concluding remarks and suggest future research.

2. The Burr type III distribution

The Burr type III distribution is the reciprocal of the Burr type XII variable. The probability density function and cumulative density function of the three-parameter Burr type III distribution are:

$$f(x; b, c, k) = \frac{kc}{b}\frac{\left(\frac{b}{x}\right)^{c+1}}{\left[1+\left(\frac{b}{x}\right)^{c}\right]^{k+1}}, \quad x \ge 0,\ b > 0,\ c > 0,\ k > 0 \tag{1}$$

and

$$F(x; b, c, k) = \frac{1}{\left[1+\left(\frac{b}{x}\right)^{c}\right]^{k}}, \quad x \ge 0,\ b > 0,\ c > 0,\ k > 0. \tag{2}$$

The scale parameter b offers the Burr type III distribution further flexibility. Each parameter has a clear statistical meaning. Parameter b is a scale parameter, and c and k are shape parameters. The density function f is unimodal at $x = b \cdot [(ck-1)/(c+1)]^{1/c}$ for $c > 1/k$ and L-shaped for $c \le 1/k$. Furthermore, the jth moment and the qth quantile of the three-parameter Burr type III distribution are given by

$$E(x^j) = b^j \frac{\Gamma\left(k+\frac{c}{j}\right) \times \Gamma\left(1-\frac{c}{j}\right)}{\Gamma(k)} \tag{3}$$

and

$$x_q = \frac{b}{\left[\left(\frac{1}{q}\right)^{1/k} - 1\right]^{1/c}}. \tag{4}$$

To ensure the existence of the jth moment, these quantiles are limited under the conditions $b > 0$, $c > 0$ and $k > 0$ and possibly $c > j$. The constrained maximum likelihood procedure was employed by Schoenberg [13] and Jamshidian and Bentler [14]. In general, the maximum likelihood estimation is a common procedure to determine the value of parameters for the log-likelihood distribution function as a maximum. The log-likelihood function of the three-parameter Burr type III distribution is:

$$\ln L = \ln\left(\prod_{i=1}^{n} f(x_i)\right) = n \cdot \ln(ck/b) + \sum_{i=1}^{n}\{(c+1)\cdot \ln[b/x_i] - (k+1)\cdot \ln[1+(b/x_i)^c]\}. \tag{5}$$

The detailed estimation procedure using the MLE method is given in the Appendix. The MLE method is very sensitive by the starting values. When $k = 1$, the Burr type III distribution becomes a log-logistic distribution. Then, we can easily obtain the initial values from a log-logistic distribution. Therefore, we can properly choose the initial values of the parameters; this will result in parameter optimization.

When sample size $n = 50$; true parameter values $b = 0.2, 2$; $c = 2, 5$; $k = 0.1, 3$. The above combination curves of pdf are showed in Fig. 1.

3. Robust methods

In this section, we present two methods (the traditional M-estimator and the AM-estimator) which are available to estimate the three parameters of the Burr type III distribution. The three unknown parameters of $(\lambda, \alpha, \beta) = \theta^T$ of the Burr type III distribution are estimations derived from n observations $(x_1, x_2, \ldots, x_n) = x^T$. They are related by

$$x_i = f_i(\theta) + u_i, \quad i = 1, \ldots, n \tag{6}$$

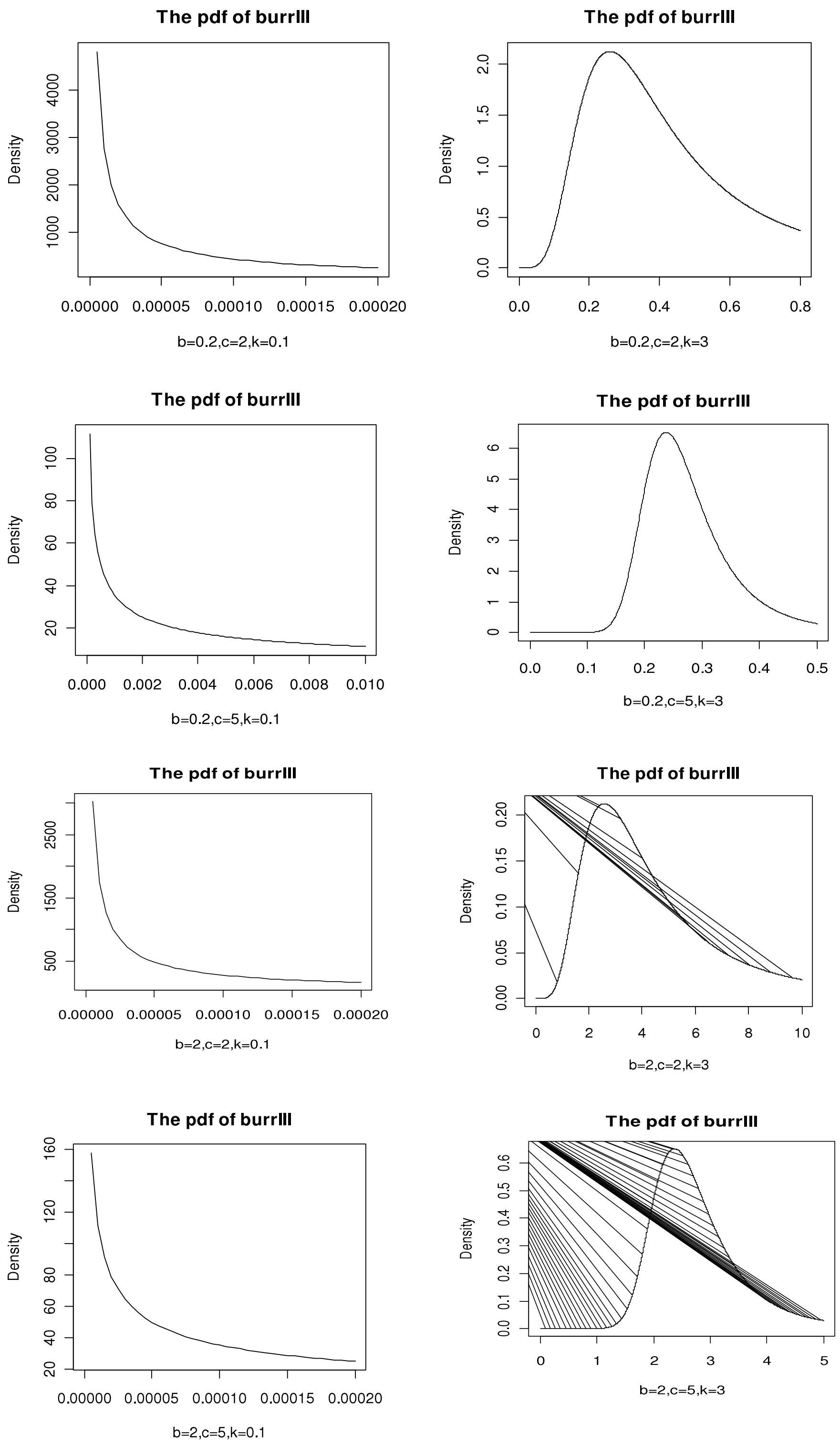

Fig. 1. Some pdf curves for the Burr type III distribution.

where $f_i(\theta)$ is the quantile function of the Burr type III distribution (see in Eq. (4)) and the error term u_i is identically independent with a mean of zero and a variance of σ^2. We can estimate the unknown true θ by a value $\hat{\theta}$ obtained from the residuals, which is given by $u_i = u_i(\theta) = x_i - f_i(\theta),\ i = 1, \ldots, n$. The approach based on the quantile function has been proved for its robust property (see [15]).

For the robust regression method, it is necessary to scale the invariant error, which is given by

$$e_i = \frac{u_i}{s} \tag{7}$$

where $s = \frac{\mathrm{MAD}(u)}{0.6745}$ and $\mathrm{MAD}(u) = \mathrm{MAD}(u_1, u_2, \ldots, u_n) = \mathrm{Median}\{|u - \mathrm{Median}(u)|\}$ (see [16]). Note that the scaling by 0.6745 is a fine tuning for the Gaussian distribution.

(1) The traditional M-estimator.

The traditional M-estimator is understood to be a weighted least squares procedure. Weight is determined by the residuals. There exists a negative correlation between residual size and its influence; the smaller the residual is, the more influential it becomes. The M-estimator method for estimating parameters of the Burr type III distribution is defined by minimizing the objective function of the invariant errors, which is given by

$$\text{Minimize} \sum_{i=1}^{n} \rho(e_i). \tag{8}$$

To estimate the three unknown parameters, we made a simple comparison study among five different objective functions (Huber's weight, Tukey's biweight, Cauchy's weight, the skipped mean, and the bisquare). The results showed that the bisquare objection is the best approach. Thus, we selected the M-estimator with the bisquare objective function as the weighted function in this study. The bisquare objective function is given by

$$\rho(e_i) = \begin{cases} \left(\dfrac{B^2}{6}\right) \cdot \left\{1 - \left[1 - \left(\dfrac{e_i}{B}\right)^2\right]^3\right\}, & \forall\, |e_i| \le B \\ \dfrac{B^2}{6}, & \forall\, |e_i| > B \end{cases}$$

$$\text{and} \quad \psi = \rho', \quad \text{which is } \psi(e_i) = \begin{cases} e_i \left(1 - (e_i/B)^2\right)^2, & \forall |e_i| \le B \\ 0, & \forall |e_i| > B \end{cases} \tag{9}$$

where $B = 4.685$ and e_i is the scale invariant error in Eq. (7).

(2) The AM-estimator.

When the data distribution is asymmetric with outliers, we propose an M-estimator with asymmetric influence function. The asymmetric bisquare objective function is given by

$$\rho(e_i) = \begin{cases} \left(\dfrac{c_1^2}{6}\right) \cdot \left\{1 - \left[1 - \left(\dfrac{e_i}{c_1}\right)^2\right]^3\right\}, & -c_1 \le e_i \le 0 \\ \left(\dfrac{c_2^2}{6}\right) \cdot \left\{1 - \left[1 - \left(\dfrac{e_i}{c_2}\right)^2\right]^3\right\}, & 0 \le e_i \le c_2 \\ \dfrac{c_1^2}{6}, & e_i \le -c_1 \\ \dfrac{c_2^2}{6}, & c_2 \le e_i \end{cases}$$

$$\text{and} \quad \psi = \rho', \quad \text{which is } \psi_{c_1,c_2}(e_i) = \begin{cases} e_i \left(1 - (e_i/c_1)^2\right)^2, & -c_1 \le e_i \le 0 \\ e_i \left(1 - (e_i/c_2)^2\right)^2, & 0 \le e_i \le c_2 \end{cases} \tag{10}$$

where $c_1 = \frac{4.685 \cdot s}{1.2}$, $c_2 = \frac{4.685 \cdot s}{0.8}$. For simplicity, the tuning parameters c_1 and c_2 are made to depend on one another. It is also clear that as data lose homogeneity, the Huber M-estimator gains precision, that is to say, in homogeneous areas it is less precise than in heterogeneous areas, becoming less precise as the proportion and magnitude of the contamination increase, because of the constant weight that the function assigns to extreme values. In contrast, AM-estimators can achieve good precision independently of the homogeneity, especially as the proportion and magnitude of the contamination increase because of the decreasing weight that the function ψ_{c_1,c_2} assigns to extreme values. For the tuning parameters, there are several pairs of values of $c = (c_1, c_2)$ with the same relative asymptotic efficiency, and the rule does not determine c uniquely.

The estimated parameters can be obtained by differentiating Eq. (10) with respect to b, c and k, respectively, and then set to zero. Then, we can obtain the simultaneous equations, which are given as follows:

$$\sum_{i=1}^{n} \psi(e_i) \frac{\partial f_i}{\partial b} = \sum_{i=1}^{n} \psi(e_i) \frac{1}{\left[\left(\frac{1}{q}\right)^{1/k} - 1\right]^{1/c}} = 0$$

$$\sum_{i=1}^{n}\psi(e_i)\frac{\partial f_i}{\partial c}=\sum_{i=1}^{n}\psi(e_i)\frac{1}{c^2}\cdot\ln\left(\left(\frac{1}{q}\right)^{1/k}-1\right)\cdot\frac{b}{\left[\left(\frac{1}{q}\right)^{1/k}-1\right]^{1/c}}=0$$

$$\sum_{i=1}^{n}\psi(e_i)\frac{\partial f_i}{\partial k}=\sum_{i=1}^{n}\psi(e_i)\cdot b\cdot\frac{-1}{c}\cdot\left(\left(\frac{1}{q}\right)^{1/k}-1\right)^{\frac{-1}{c}-1}\cdot\frac{-1}{k^2}\cdot\left[\ln\left(\frac{1}{q}\right)\right]\cdot\left(\frac{1}{q}\right)^{\frac{1}{k}}=0.$$

In order to solve the above equations, the Newton–Raphson method can be employed. Again, we can apply the initial values by the MLE method in the previous section. Thus, it will result in parameter optimization.

For the M-estimator's normal approximation, we have to check all regularity conditions for the Burr type III distribution (see [17]) to check it. For the Burr type III distribution, we have that (1) the first and second derivative functions exist and satisfy the continuous property; (2) the expectation of the second derivative function is less than infinite; (3) the function is a continuous function and is non-null; (4) the space of parameters is a closed, bounded subset of R^p; (5) $F=E\left[\left(\frac{\mathrm{d}\psi(x,\theta)}{\mathrm{d}\theta}\right)\right]$ exists and is nonsingular; and (6) the ε_i are i.i.d. with mean zero and variance σ^2 ($\sigma^2>0$). Thus, we can conclude that all regularity conditions are satisfied for the Burr type III distribution. In addition, we can find a robust M-estimator to minimize an appropriate loss function. Such a function is chosen to down weight the effects upon the fit of observations with large residuals. Thus, θ is chosen to minimize a loss function of the form $h(\theta)=\sum_{i=1}^{n}\rho\left(\frac{r_i(\theta)}{\sigma}\right)$, where σ is some measure of dispersion. We shall deal only with methods for minimizing a function of the form $h(\theta)$ which can be used when some of the $r_i's$ are nonlinear (see [16,10,17,18]).

Thus, an approximate $(1-\alpha)\%$ confidence interval on $\theta^T=(b,c,k)$ can be obtained by

$$\sqrt{n}\left(\hat{\theta}-\theta\right)\underset{n\to\infty}{\overset{d}{\to}}N(0,C^{-1}AC^{-1'}) \tag{11}$$

where $A_{ij}=E[\psi_i(e)\cdot\psi_j(e)]$, $C_{ij}=\left.\frac{\partial\lambda_i(\theta)}{\partial\theta_j}\right|_{\theta=\hat{\theta}}$ and $\lambda_i(\theta)=E[\psi_i(e)]\forall\, i=1,2,3$. Under general assumptions (see [8]), we can obtain the following:

$$\psi_1(e_i)=\psi(e_i)\frac{\partial f_i}{\partial b}=\psi(e_i)\cdot\frac{1}{\left[\left(\frac{1}{q}\right)^{1/k}-1\right]^{1/c}}$$

$$\psi_2(e_i)=\psi(e_i)\cdot\frac{\partial f_i}{\partial c}=\psi(e_i)\cdot\frac{1}{c^2}\cdot\ln\left[\left(\frac{1}{q}\right)^{1/k}-1\right]\cdot\frac{b}{\left[\left(\frac{1}{q}\right)^{1/k}-1\right]^{1/c}}$$

$$\psi_3(e_i)=\psi(e_i)\cdot\frac{\partial f_i}{\partial k}=\psi(e_i)\cdot b\cdot\frac{-1}{c}\cdot\left[\left(\frac{1}{q}\right)^{1/k}-1\right]^{\frac{-1}{c}-1}\cdot\frac{-1}{k^2}\cdot\left[\ln\left(\frac{1}{q}\right)\right]\cdot\left(\frac{1}{q}\right)^{\frac{1}{k}}.$$

In addition, we have

$$\frac{\partial\lambda_1(\theta)}{\partial b}=\frac{1}{n}\sum_{i=1}^{n}\psi'(e_i)\left(\frac{-1}{s}\right)\cdot\frac{\partial f_i}{\partial b}\cdot\frac{\partial f_i}{\partial b}=\frac{1}{n}\sum_{i=1}^{n}\psi'(e_i)\left(\frac{-1}{s}\right)\cdot\left(\frac{1}{\left[\left(\frac{1}{q}\right)^{1/k}-1\right]^{1/c}}\right)^2$$

$$\frac{\partial\lambda_1(\theta)}{\partial c}=\frac{1}{n}\sum_{i=1}^{n}\psi'(e_i)\left(\frac{-1}{s}\right)\cdot\frac{\partial f_i}{\partial c}\cdot\frac{\partial f_i}{\partial b}+\frac{1}{n}\sum_{i=1}^{n}\psi(e_i)\cdot\frac{\partial f_i}{\partial b}\cdot\frac{\ln((1/q)^{1/k}-1)}{c^2}$$

$$\frac{\partial\lambda_1(\theta)}{\partial k}=\frac{1}{n}\sum_{i=1}^{n}\psi'(e_i)\left(\frac{-1}{s}\right)\cdot\frac{\partial f_i}{\partial k}\cdot\frac{\partial f_i}{\partial b}+\frac{1}{n}\sum_{i=1}^{n}\psi(e_i)\cdot\frac{\partial f_i}{\partial b}\cdot\frac{1}{c}\cdot\left[\left(\frac{1}{q}\right)^{1/k}-1\right]^{\frac{1}{c}-1}\cdot\frac{-1}{k^2}\cdot\left[\ln\left(\frac{1}{q}\right)\right]\cdot\left(\frac{1}{q}\right)^{\frac{1}{k}}$$

$$\frac{\partial\lambda_2(\theta)}{\partial b}=\frac{1}{n}\sum_{i=1}^{n}\psi'(e_i)\left(\frac{-1}{s}\right)\cdot\frac{\partial f_i}{\partial b}\cdot\frac{\partial f_i}{\partial c}+\frac{1}{n}\sum_{i=1}^{n}\psi(e_i)\cdot\frac{\partial}{\partial b}\left(\frac{\partial f_i}{\partial c}\right)$$

$$\frac{\partial\lambda_2(\theta)}{\partial c}=\frac{1}{n}\sum_{i=1}^{n}\psi'(e_i)\left(\frac{-1}{s}\right)\cdot\frac{\partial f_i}{\partial c}\cdot\frac{\partial f_i}{\partial c}+\frac{1}{n}\sum_{i=1}^{n}\psi(e_i)\cdot\frac{\partial}{\partial c}\left(\frac{\partial f_i}{\partial c}\right)$$

$$\frac{\partial\lambda_2(\theta)}{\partial k}=\frac{1}{n}\sum_{i=1}^{n}\psi'(e_i)\left(\frac{-1}{s}\right)\cdot\frac{\partial f_i}{\partial k}\cdot\frac{\partial f_i}{\partial c}+\frac{1}{n}\sum_{i=1}^{n}\psi(e_i)\cdot\frac{\partial}{\partial k}\left(\frac{\partial f_i}{\partial c}\right)$$

$$\frac{\partial\lambda_3(\theta)}{\partial b} = \frac{1}{n}\sum_{i=1}^{n}\psi'(e_i)\left(\frac{-1}{s}\right)\cdot\frac{\partial f_i}{\partial b}\cdot\frac{\partial f_i}{\partial k} + \frac{1}{n}\sum_{i=1}^{n}\psi(e_i)\cdot\frac{\partial}{\partial b}\left(\frac{\partial f_i}{\partial k}\right)$$

$$\frac{\partial\lambda_3(\theta)}{\partial c} = \frac{1}{n}\sum_{i=1}^{n}\psi'(e_i)\left(\frac{-1}{s}\right)\cdot\frac{\partial f_i}{\partial c}\cdot\frac{\partial f_i}{\partial k} + \frac{1}{n}\sum_{i=1}^{n}\psi(e_i)\cdot\frac{\partial}{\partial c}\left(\frac{\partial f_i}{\partial k}\right)$$

$$\frac{\partial\lambda_3(\theta)}{\partial k} = \frac{1}{n}\sum_{i=1}^{n}\psi'(e_i)\left(\frac{-1}{s}\right)\cdot\frac{\partial f_i}{\partial k}\cdot\frac{\partial f_i}{\partial k} + \frac{1}{n}\sum_{i=1}^{n}\psi(e_i)\cdot\frac{\partial}{\partial k}\left(\frac{\partial f_i}{\partial k}\right)$$

where $\frac{\partial^2 f_i}{\partial b\partial c} = \frac{\ln\left[\left(\frac{1}{q}\right)^{1/k}-1\right]}{\left[\left(\frac{1}{q}\right)^{1/k}-1\right]^{1/c}\cdot c^2}$

$$\frac{\partial^2 f_i}{\partial b\partial k} = \frac{\left[\left(\frac{1}{q}\right)^{1/k}-1\right]^{-(1+c)/c}\left(\frac{1}{q}\right)^{1/k}\ln\left(\frac{1}{q}\right)}{c\cdot k^2}$$

$$\frac{\partial^2 f_i}{\partial c\partial k} = \frac{b\left[\left(\frac{1}{q}\right)^{1/k}-1\right]^{-(1+c)/c}\left(\frac{1}{q}\right)^{1/k}\ln\left(\frac{1}{q}\right)\left\{\ln\left[\left(\frac{1}{q}\right)^{1/k}-1\right]-c\right\}}{c^3\cdot k^2}$$

$$\frac{\partial^2 f_i}{\partial b^2} = 0$$

$$\frac{\partial^2 f_i}{\partial c^2} = \frac{b\ln\left[\left(\frac{1}{q}\right)^{1/k}-1\right]^{-1/c}\ln\left[\left(\frac{1}{q}\right)^{1/k}-1\right]\left\{\ln\left[\left(\frac{1}{q}\right)^{1/k}-1\right]-2c\right\}}{c^4}$$

$$\begin{aligned}\frac{\partial^2 f_i}{\partial k^2} = {} & \frac{b\left[\left(\frac{1}{q}\right)^{1/k}\right]^2\ln\left(\frac{1}{q}\right)^2}{\left[\left(\frac{1}{q}\right)^{1/k}-1\right]^{1/c}c^2k^4\left[\left(\frac{1}{q}\right)^{1/k}-1\right]^2} - \frac{b\left(\frac{1}{q}\right)^{1/k}\ln\left(\frac{1}{q}\right)^2}{\left[\left(\frac{1}{q}\right)^{1/k}-1\right]^{1/c}ck^4\left[\left(\frac{1}{q}\right)^{1/k}-1\right]} \\ & - \frac{b\left[\left(\frac{1}{q}\right)^{1/k}-1\right]^{-1/c}\ln\left[\left(\frac{1}{q}\right)^{1/k}-1\right]\left\{\left[\left(\frac{1}{q}\right)^{1/k}-1\right]-2c\right\}}{\left[\left(\frac{1}{q}\right)^{1/k}-1\right]^{1/c}ck^3\left[\left(\frac{1}{q}\right)^{1/k}-1\right]} + \frac{b\left[\left(\frac{1}{q}\right)^{1/k}\right]^{2^2}\ln\left(\frac{1}{q}\right)^2}{\left[\left(\frac{1}{q}\right)^{1/k}-1\right]^{1/c}ck^4\left[\left(\frac{1}{q}\right)^{1/k}-1\right]^2}.\end{aligned}$$

4. Simulation study

In order to assess the performance of the MLE, the traditional M-estimator and the AM-estimator, we applied the Monte Carlo simulation. Complete samples with outliers are randomly generated from the three-parameter Burr type III distribution with the specified values of b, c and k. The procedures for generating samples are given as follows:

Step 1. Generate n random samples $y_1, y_2, \ldots, y_n$ from uniform distribution $\sim U(0, 1)$.

Step 2. Obtain the inverse function of the cumulative density function from Eq. (2). Then, for the specified values of b, c and k, the random sample x_i for the Burr type III distribution can be obtained by $x_i = b\cdot(y_i^{-1/k}-1)^{-1/c}\ \forall i = 1, 2, \ldots, n$.

Step 3. Outliers are generated from a random sample from uniform distribution $\sim U(\bar{X}+4S, \bar{X}+7S)$, where $\bar{X}$ is the sample mean of $X = (x_1, x_2, \ldots, x_n)$ and S is the sample standard deviation of X (see [19]).

The simulation study included the following conditions: sample size $n = 20, 50$; true parameter values $b = 0.2, 2$; $c = 2, 5$; $k = 0.1, 3$; the number of outliers = 1%, 2% and 20%. Also, each simulation condition was generated by 3000 replications. We choose the distribution that is asymmetric. The simulation program was run in S-Plus software. The request of computer program in R language is available to authors. Here, two measures, the bias and root mean square error (RMSE), are used to assess the performance of three different methods. They are given by

$$\widehat{\mathrm{Bias}}(\hat{b}) = \bar{b} - b, \qquad \widehat{\mathrm{Bias}}(\hat{c}) = \bar{c} - c, \qquad \widehat{\mathrm{Bias}}(\hat{k}) = \bar{k} - k$$

and

$$\widehat{\mathrm{RMSE}}(\hat{b}) = \sqrt{\left(\frac{1}{N}\right)\cdot\sum_{i=1}^{N}(\hat{b}_i - b)^2}, \qquad \widehat{\mathrm{RMSE}}(\hat{c}) = \sqrt{\left(\frac{1}{N}\right)\cdot\sum_{i=1}^{N}(\hat{c}_i - c)^2},$$

Table 1
$\widehat{\text{Bias}}$ and $\widehat{\text{RMSE}}$ (parentheses) of the three estimator for the sample size n with one outlier.

n	b	c	k	Parameter ($\hat{b}$)			Parameter ($\hat{c}$)			Parameter ($\hat{k}$)		
				MLE	M-estimator	AM-estimator	MLE	M-estimator	AM-estimator	MLE	M-estimator	AM-estimator
20	0.2	2	0.1	**0.0066** (**0.1263**)	0.2225 (0.6725)	0.0469 (1.5194)	12.2098 (18.5754)	−0.1664 (0.6204)	**0.0067** (**0.0480**)	−0.0389 (0.0946)	0.0141 (0.0488)	**0.0003** (**0.0019**)
20	0.2	2	3	0.0765 (0.1576)	0.0143 (0.4531)	**−0.0001** (**0.0147**)	0.1186 (0.5427)	0.0045 (0.1207)	**0.0120** (**0.1322**)	−0.0433 (2.5895)	0.0969 (7.5170)	**0.0022** (**0.1287**)
20	0.2	5	0.1	**0.0039** (**0.0576**)	0.0114 (0.1525)	0.0114 (0.0898)	−0.5614 (7.1124)	−0.0445 (1.4969)	**0.0053** (**0.1271**)	0.0710 (0.2104)	0.0021 (0.0343)	**0.0003** (**0.0035**)
20	0.2	5	3	−0.0149 (0.0508)	0.0027 (0.0347)	**0.0005** (**0.0075**)	−0.3767 (1.1624)	0.0367 (0.2528)	**−0.0035** (**0.1906**)	3.5779 (7.7512)	−0.0158 (0.3245)	**0.0006** (**0.0621**)
20	2	2	0.1	**0.0690** (**1.1518**)	0.0829 (0.4737)	0.0718 (0.7349)	9.7202 (16.6334)	−0.1658 (0.5857)	**−0.0149** (**0.1792**)	−0.0097 (0.1900)	0.0168 (0.0727)	**0.0012** (**0.0139**)
20	2	2	3	0.5540 (1.5050)	−0.2021 (0.4347)	**−0.1678** (**0.3564**)	0.1219 (0.5568)	−0.0357 (0.3086)	**−0.0608** (**0.1908**)	1.0920 (4.7294)	0.9501 (2.5414)	**0.5637** (**1.3701**)
20	2	5	0.1	**0.0017** (**0.5543**)	−0.0311 (0.1720)	0.0028 (0.3430)	−1.5103 (3.8143)	−0.1767 (10.5937)	**−0.0116** (12.0031)	0.0864 (0.1794)	0.0050 (0.1050)	**0.0044** (**0.1306**)
20	2	5	3	**−0.0496** (**0.4751**)	−0.1270 (0.2602)	−0.0596 (0.1759)	−0.2709 (1.2080)	**−0.0064** (**0.9450**)	−0.1039 (0.4688)	1.9341 (4.9674)	1.6206 (3.5027)	**0.5409** (**1.8273**)
50	0.2	2	0.1	**0.0251** (**0.0907**)	0.1240 (0.4600)	0.0479 (2.1163)	4.0124 (9.4225)	−0.0297 (0.5459)	**0.0033** (**0.0414**)	−0.0265 (0.0568)	0.0086 (0.0441)	**0.0001** (**0.0015**)
50	0.2	2	3	0.0281 (0.1036)	0.0176 (0.1127)	**−0.0002** (**0.0124**)	0.0480 (0.3530)	0.0020 (0.0977)	**0.0037** (**0.0712**)	0.5493 (2.7374)	−0.0563 (0.6292)	**−0.0045** (**0.1158**)
50	0.2	5	0.1	−0.0033 (0.0343)	0.0030 (0.1021)	**0.0034** (**0.0236**)	−0.9944 (2.5754)	0.0924 (1.2389)	**−0.0104** (**0.1492**)	0.0370 (0.0602)	−0.0021 (0.0253)	**0.0002** (**0.0230**)
50	0.2	5	3	−0.0147 (0.0431)	0.0016 (0.0283)	**0.0002** (**0.0034**)	−0.2300 (0.7874)	0.0123 (0.1535)	**−0.0032** (**0.1622**)	3.0621 (6.9078)	−0.0115 (0.2241)	**0.0031** (**0.0561**)
50	2	2	0.1	0.1502 (0.8923)	0.0595 (0.3466)	**0.0620** (**0.7078**)	4.1295 (9.6834)	−0.1146 (0.5422)	**−0.0202** (**0.1685**)	−0.0213 (0.0581)	0.0139 (0.0612)	**0.0009** (**0.0112**)
50	2	2	3	0.2331 (1.0753)	**−0.1438** (**0.3386**)	−0.1593 (0.3299)	0.0383 (0.3623)	−0.0202 (0.2173)	**−0.0572** (**0.1579**)	1.0775 (3.9820)	0.6551 (1.6640)	**0.5575** (**1.2799**)
50	2	5	0.1	−0.0364 (0.3496)	−0.0234 (0.1286)	**−0.0071** (**0.2262**)	−1.0317 (2.0031)	**−0.0653** (**1.1209**)	−0.1286 (0.5492)	0.0391 (0.0713)	0.0021 (0.0270)	**0.0029** (**0.0202**)
50	2	5	3	−0.0630 (0.3891)	−0.0685 (0.1655)	**−0.0550** (**0.1524**)	−0.1040 (0.8149)	**0.0233** (**0.6070**)	−0.1050 (0.3455)	1.6530 (4.2054)	0.9786 (2.5129)	**0.5259** (**1.6887**)

$$\widehat{\text{RMSE}}(\hat{b}) = \sqrt{\left(\frac{1}{N}\right) \cdot \sum_{i=1}^{N} (\hat{k}_i - k)^2}$$

where $\overline{b} = \frac{1}{N}\sum_{i=1}^{N} \hat{b}_i$, $\overline{c} = \frac{1}{N}\sum_{i=1}^{N} \hat{c}_i$, $\overline{k} = \frac{1}{N}\sum_{i=1}^{N} \hat{k}_i$.

The simulation results for complete data with one and two outliers are presented in Tables 1–3. The following conclusions from the simulation study were observed:

(1) The presence of outliers in the data led to either an underestimation or overestimation of the parameters for all possible combinations.
(2) For complete data with only one outlier, we found that the bias of all three parameters using the AM-estimator method outperformed the other two methods (the M-estimator and the MLE) in all cases. We also found that the sample size did not affect the bias of the M-estimator and AM-estimator methods in most cases. However, the sample size did affect the bias of the MLE method in most cases.
(3) With respect to the RMSE, we found that the AM-estimator method outperformed the other two methods (the M-estimator and the MLE) in all cases. However, the sample size affected the RMSE of all three methods in most cases. That is, when the sample size is increased, the RMSE value would decrease.
(4) The simulation results for complete data with two outliers and 20% outliers are very similar to the results of complete data with one outlier.

5. Numerical example

We consider a real sample data set with sample size n=36 for chromium in marine waters, which is identified as the three-parameter Burr type III distribution (see [5]). The data is given as follows:

Table 2
$\widehat{\text{Bias}}$ and $\widehat{\text{RMSE}}$ (parentheses) of the three estimator for the sample size n with two outliers.

n	b	c	k	Parameter ($\hat{b}$)			Parameter ($\hat{c}$)			Parameter ($\hat{k}$)		
				MLE	M-estimator	AM-estimator	MLE	M-estimator	AM-estimator	MLE	M-estimator	AM-estimator
20	0.2	2	0.1	**0.0058** (**0.1113**)	0.1714 (0.6010)	0.0535 (1.6421)	12.2991 (18.8017)	−0.1665 (0.5920)	**0.0095** (**0.0507**)	−0.0467 (0.0804)	0.0115 (0.0426)	**0.0004** (**0.0021**)
		2	3	0.0428 (0.1296)	0.0101 (0.0885)	**0.0012** (**0.0247**)	−0.1344 (0.3920)	0.0124 (0.1301)	**0.0121** (**0.2706**)	0.5426 (3.0767)	−0.0430 (0.5498)	**0.0013** (**0.1364**)
		5	0.1	0.0280 (0.0765)	−0.0053 (0.1448)	**0.0145** (**0.0656**)	1.3733 (12.5640)	−0.2629 (1.5331)	**0.0129** (**0.1324**)	0.0938 (0.3202)	0.0024 (0.0378)	**0.0004** (**0.0036**)
		5	3	−0.0319 (0.0552)	0.0029 (0.0241)	**0.0008** (**0.0085**)	−0.9343 (1.2554)	0.0431 (0.2564)	**−0.0160** (**0.2580**)	5.0771 (9.5970)	−0.0311 (0.3436)	**0.0077** (**0.1183**)
20	2	2	0.1	0.2333 (1.2371)	0.0657 (0.3856)	**0.0562** (**0.5130**)	11.1111 (17.9031)	−0.2253 (0.5845)	**−0.0132** (**0.1912**)	−0.0309 (0.1468)	0.0169 (0.0788)	**0.0005** (**0.0077**)
		2	3	0.2745 (1.3435)	−0.2133 (0.4240)	**−0.1944** (**0.3895**)	−0.1192 (0.3904)	**0.0067** (**0.3702**)	−0.0699 (0.1865)	1.8609 (5.4231)	0.9931 (2.6753)	**0.6083** (**1.4339**)
		5	0.1	0.2405 (0.7229)	−0.0571 (0.1896)	**−0.0015** (**0.2424**)	−1.4018 (6.6551)	**−0.5187** (**1.6398**)	−0.0822 (0.4927)	0.1169 (0.2058)	0.0027 (0.0318)	**0.0017** (**0.0180**)
		5	3	−0.2475 (0.5133)	−0.1652 (0.2992)	**−0.0575** (**0.1752**)	−0.9283 (1.2746)	0.1692 (1.3745)	**−0.1078** (**0.4520**)	3.4515 (6.4645)	2.1694 (4.0309)	**0.5325** (**1.7952**)
50	0.2	2	0.1	0.0286 (0.0868)	0.1104 (0.3951)	**0.0127** (**0.0439**)	4.2171 (9.5574)	−0.0458 (0.4888)	**0.0041** (**0.0381**)	−0.0316 (0.0529)	0.0065 (0.0426)	**0.0002** (**0.0010**)
		2	3	0.0104 (0.0929)	0.0221 (0.1227)	**0.0007** (**0.0251**)	−0.0669 (0.2956)	0.0120 (0.1097)	**0.0042** (**0.1087**)	0.9397 (3.1120)	−0.0942 (0.6063)	**−0.0042** (**0.1206**)
		5	0.1	0.0008 (0.0363)	−0.0048 (0.1146)	**0.0056** (**0.0257**)	−1.5808 (3.4237)	0.1034 (1.3572)	**−0.0075** (**0.1531**)	0.0631 (0.0815)	−0.0037 (0.0290)	**0.0000** (**0.0032**)
		5	3	−0.0279 (0.0480)	0.0019 (0.0253)	**0.0004** (**0.0050**)	−0.6055 (0.8610)	0.0254 (0.1718)	**−0.0008** (**0.1711**)	4.4108 (8.3677)	−0.0171 (0.2408)	**0.0024** (**0.0634**)
50	2	2	0.1	0.2337 (0.8971)	**0.0435** (**0.3214**)	0.0576 (0.7169)	4.3513 (9.9247)	−0.1315 (0.5247)	**−0.0163** (**0.1612**)	−0.0263 (0.0576)	0.0117 (0.0544)	**0.0005** (**0.0087**)
		2	3	**0.0422** (**0.9704**)	−0.1675 (0.3557)	−0.1701 (0.3389)	−0.0778 (0.3013)	−0.0036 (0.2545)	**−0.0611** (**0.1588**)	1.6158 (4.4754)	0.7650 (1.7286)	**0.5861** (**1.2917**)
		5	0.1	**0.0013** (**0.3587**)	−0.0438 (0.1301)	−0.0111 (0.1615)	−1.6804 (2.7366)	−0.0879 (1.2103)	**−0.1126** (**0.5113**)	0.0652 (0.0850)	0.0021 (0.0276)	**0.0026** (**0.0202**)
		5	3	−0.2107 (0.4264)	−0.0870 (0.1810)	**−0.0567** (**0.1612**)	−0.5322 (0.8379)	0.1238 (0.7543)	**−0.1179** (**0.3675**)	2.8821 (5.5278)	1.2875 (2.7981)	0.5467 (1.7947)

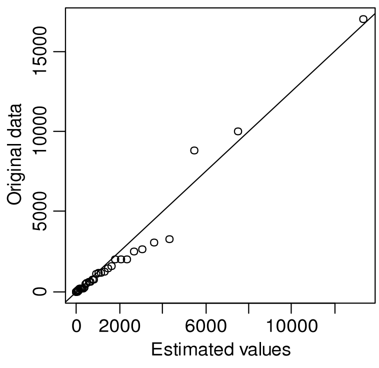

Fig. 2. The Q–Q plot of the MLE-estimator for example.

2.40, 4.00, 4.79, 9.55, 12.59, 39.81, 56.00, 89.13, 140.00, 177.83, 187.20, 199.50, 199.53, 210.00, 263.03, 478.63, 540.00, 602.56, 602.56, 728.00, 776.25, 1122.02, 1174.89, 1200.00, 1258.93, 1456.00, 1600.00, 1995.26, 2000.00, 2000.00, 2511.89, 2630.27, 3090.30, 3311.31, 8800.00, 10 000.00.

To determine the more efficient approach in determining outliers, we compared the estimators of the MLE, the M-estimator and the AM-estimator. First, we estimated the parameters using the MLE method and the estimated values of (b, c, k) are (2599.2, 2.065, 0.2276). Now, we treat the 3rd data point as an outlier; its value becomes 17 000. We estimated the parameters using the MLE method and the estimated values of (b, c, k) are (2599.19, 1.6560, 0.3008). Fig. 2 demonstrates how outliers could potentially distort the parameter estimation of MLE. Using the M-estimator method, the estimated values

Table 3
$\widehat{\text{Bias}}$ and $\widehat{\text{RMSE}}$ (parentheses) of the three estimator for the sample size n with 20% outliers.

n	b	c	k	Parameter (b)			Parameter (c)			Parameter (k)		
				MLE	M-estimator	AM-estimator	MLE	M-estimator	AM-estimator	MLE	M-estimator	AM-estimator
20	0.2	2	0.1	**0.0058** (**0.1060**)	0.1715 (0.4593)	0.0424 (0.1030)	11.2529 (17.9815)	−0.2849 (0.6356)	**0.0158** (**0.0796**)	−0.0505 (0.0647)	0.0074 (0.0366)	**0.0007** (**0.0019**)
		2	3	0.0315 (0.1268)	0.0075 (0.0835)	0.0024 (0.0416)	−0.3237 (0.4405)	0.0147 (0.1450)	**0.0109** (**0.1802**)	0.7985 (3.0261)	0.0201 (3.4669)	**−0.0027** (**0.1276**)
		5	0.1	0.0960 (0.1403)	−0.0029 (0.1457)	**0.0225** (**0.1998**)	31.2976 (39.6712)	−0.2843 (1.3587)	**−0.0068** (**0.2187**)	−0.0463 (0.1092)	0.0113 (0.0639)	**0.0001** (**0.0041**)
		5	3	−0.0476 (0.0653)	0.0039 (0.0296)	**0.0011** (**0.0102**)	−1.4883 (1.6694)	0.0928 (0.4486)	**−0.0075** (**0.6392**)	6.8394 (11.3335)	−0.0454 (0.3749)	**0.0096** (**0.0891**)
20	2	2	0.1	0.5784 (1.5077)	0.0708 (0.2952)	**0.0449** (**0.5833**)	12.4956 (19.6357)	−0.2703 (0.5433)	**−0.0281** (**0.2457**)	−0.0460 (0.0819)	0.0036 (0.0434)	**0.0005** (**0.0100**)
		2	3	**0.0926** (**1.3006**)	−0.2278 (0.4356)	−0.3197 (0.5439)	−0.3122 (0.4274)	**−0.0081** (**0.3607**)	−0.1318 (0.2437)	2.5149 (5.8558)	0.8956 (2.1608)	**0.8431** (**1.7951**)
		5	0.1	1.0574 (1.6709)	−0.1108 (0.2780)	**0.0132** (**0.4378**)	13.6997 (25.9542)	−0.6966 (1.6449)	**−0.1625** (**0.6918**)	0.0575 (0.2292)	0.0081 (0.0688)	**0.0029** (**0.0191**)
		5	3	−0.3875 (0.5953)	−0.2310 (0.3677)	**−0.0524** (**0.1890**)	−1.5002 (1.6644)	0.2678 (1.6787)	**−0.1015** (**0.4415**)	4.4615 (7.1297)	2.8455 (4.8390)	**0.4800** (**1.9294**)
50	0.2	2	0.1	0.0289 (0.0806)	0.2302 (0.4239)	**0.0221** (**0.0620**)	3.3416 (7.4698)	−0.3430 (0.6970)	**0.0063** (**0.0937**)	−0.0384 (0.0461)	−0.0008 (0.0387)	**0.0004** (**0.0016**)
		2	3	−0.0385 (0.0915)	0.0031 (0.0737)	**0.0020** (**0.0319**)	−0.4692 (0.4993)	0.0076 (0.1169)	**0.0080** (**0.1682**)	2.8435 (5.3924)	−0.0469 (0.6445)	**−0.0062** (**0.1194**)
		5	0.1	0.1846 (0.2066)	0.0643 (0.3380)	**0.0130** (**0.0533**)	43.6104 (48.2665)	−0.3652 (1.4993)	**−0.0174** (**0.2557**)	−0.0818 (0.0911)	0.0283 (0.0742)	**0.0001** (**0.0074**)
		5	3	−0.0731 (0.0793)	0.0063 (0.0352)	**0.0007** (**0.0102**)	−1.7931 (1.8259)	0.1147 (0.5637)	**−0.0177** (**0.4101**)	9.8570 (13.1120)	−0.0739 (0.4261)	**0.0100** (**0.1536**)
50	2	2	0.1	0.5927 (1.0516)	0.0884 (0.2075)	**−0.0002** (**0.2206**)	5.0814 (9.9172)	−0.2367 (0.4789)	**−0.0272** (**0.2041**)	−0.0428 (0.0514)	−0.0050 (0.0472)	**−0.0001** (**0.0049**)
		2	3	−0.5072 (0.9965)	−0.2143 (0.4205)	**−0.3168** (**0.5347**)	−0.4706 (0.5011)	**0.0017** (**0.3162**)	−0.1353 (0.2278)	4.3116 (7.4485)	0.8703 (2.1256)	**0.7262** (**1.3784**)
		5	0.1	1.6153 (1.9025)	−0.2751 (0.4241)	**−0.0142** (**0.0743**)	34.5016 (41.6072)	−0.7631 (1.7593)	**−0.0773** (**0.4888**)	−0.0559 (0.1197)	0.0103 (0.0452)	**0.0016** (**0.0137**)
		5	3	−0.7703 (0.8311)	−0.1990 (0.2716)	**−0.0451** (**0.1671**)	−1.7874 (1.8181)	0.5417 (1.5636)	**−0.1271** (**0.4019**)	11.1368 (13.8092)	2.7646 (4.0962)	**0.4076** (**1.6386**)

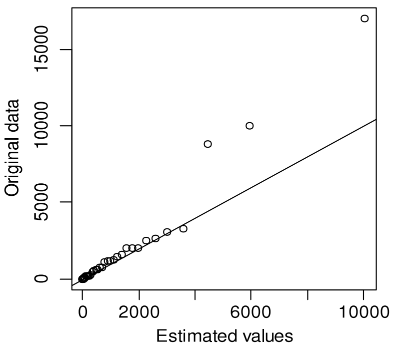

Fig. 3. The Q–Q plot of the M-estimator for example.

of (b, c, k) are (2599.14, 1.8421, 0.2436). Using the AM-estimator method, the estimated values of (b, c, k) are (2599.23, 2.0514, 0.2072).

In Fig. 2, we can see the straight line degree which moves by the outliers. However, both straight lines in Figs. 3 and 4 are not affected by the outliers. Therefore, the M-estimator and the AM-estimator are useful in reducing the effect of the outliers. In addition, the 95% confidence intervals of (b, c, k) for the MLE, the M-estimator and the AM-estimator are presented in Table 4. We may focus on the 5th and 95th percentiles from the data set. Here, we have $x_{0.05} = 6.353235$ and $x_{0.95} = 7178.925$ for the MLE method, and $x_{0.05} = 3.277225$ and $x_{0.95} = 5713.211$ for the M-estimator method. $x_{0.05} = 2.259220$ and $x_{0.95} = 4826.901$ for the AM-estimator method.

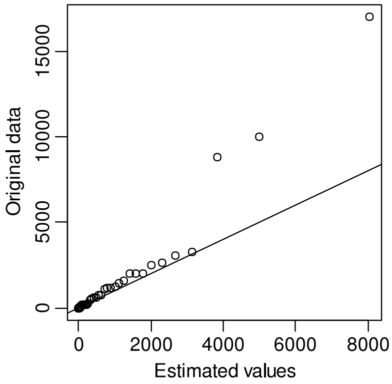

Fig. 4. The Q–Q plot of the AM-estimator for example.

Table 4
The results of parameters b, c, k for the MLE, the M-estimator and the AM-estimator for example.

Parameters	Method		
	MLE	M-estimator	AM-estimator
$\hat{b}$ (Standard deviation)	2599.19 (7765.83)	2599.19 (1112.516)	2599.23 (1070.844)
[95% Confidence intervals]	[0, 17 820.22]	[418.6595, 4779.721]	[500.3764, 4698.084]
$\hat{c}$(Standard deviation)	1.656 (3.281447)	1.8421 (0.1978302)	2.0514 (0.57905)
[95% Confidence intervals]	[0, 8.087636]	[1.454353, 2.229847]	[0.916457, 3.18634]
$\hat{k}$ (Standard deviation)	0.3008 (0.86493)	0.2436 (0.02618692)	0.2072 (0.056518)
[95% Confidence intervals]	[0, 1.996075]	[0.1922736, 0.2949264]	[0.096422, 0.317977]

Note: The parameters using the MLE method with no outlier and the estimated values of (b, c, k) are (2599.2, 2.065, 0.2276).

6. Conclusions

In this paper, we present an AM-estimator with asymmetric influence function based on the quantile function of the Burr type III distribution to estimate the parameters for complete data with outliers. The simulation results show that the AM-estimator method outperforms the other two methods (the MLE and the traditional M-estimator) in terms of the bias and root mean square error. One numerical example and one simulated data also confirm that the AM-estimator method outperforms the other two methods (the MLE and the traditional M-estimator). We may conclude that the AM-estimator with asymmetric influence function is a suitable approach to estimate the parameters of the three-parameter Burr type III distribution for complete data with outliers.

Future investigation should include other robust methods such as least trimmed squares (LTS), least median of squares (LMS) and MM-estimation, as well as the use of additional outlier data configuration via the Monte Carlo simulation study. There are other distributions, like the extreme value distribution which are suitable for the fitting of data with outliers. To compare the estimation results of the AM-estimator for extreme value distribution parameters with the results for the Burr type III distribution could be a research topic.

Acknowledgments

The authors gratefully acknowledge the referees of this paper who helped to clarify and improve this presentation.

Appendix

I. The estimated parameters using the MLE method can be obtained by differentiating Eq. (5) with respect to b, c and k, respectively and set to zero. Thus, we have

$$\frac{\partial \ln L}{\partial b} = \frac{-n}{b} + \sum_{i=1}^{n}\left[\frac{c+1}{b} - (k+1)\cdot\frac{c\cdot b^{c-1}\cdot x_i^{-c}}{1+\left(\frac{b}{x_i}\right)^c}\right] = 0$$

$$\frac{\partial \ln L}{\partial c} = \frac{n}{c} + \sum_{i=1}^{n}\left[\ln\left(\frac{b}{x_i}\right) - (k+1)\cdot\frac{\left(\frac{b}{x_i}\right)^c\cdot\ln\left(\frac{b}{x_i}\right)}{1+\left(\frac{b}{x_i}\right)^c}\right] = 0$$

$$\frac{\partial \ln L}{\partial k} = \frac{n}{k} + \sum_{i=1}^{n}\left[-\ln\left(1+\left(\frac{b}{x_i}\right)^c\right)\right] = 0.$$

Then, we can obtain the simultaneous equations which are given as follows:

$$b = \frac{n}{\sum_{i=1}^{n}\left[\frac{c+1}{b} - (k+1)\cdot\frac{c\cdot b^{c-1}\cdot x_i^{-c}}{1+\left(\frac{b}{x_i}\right)^c}\right]}$$

$$c = \frac{-n}{\sum_{i=1}^{n}\left[\ln\left(\frac{b}{x_i}\right) - (k+1)\cdot\frac{\left(\frac{b}{x_i}\right)^c\cdot\ln\left(\frac{b}{x_i}\right)}{1+\left(\frac{b}{x_i}\right)^c}\right]}$$

$$k = \frac{-n}{-\sum_{i=1}^{n}\ln\left[1+\left(\frac{b}{x_i}\right)^c\right]}.$$

In order to solve the above equations, the Newton–Raphson method can be employed.

II. For the MLE's normal approximation, we can use the theorem from Roussas [17] to check it. From the Burr type III distribution, we have that (1) the first and second derivative functions exist and satisfy the continuous property; (2) the expectation of the second derivative function is less than infinite; (3) function is a continuous function and is non-null; (4) the space of parameters is a closed, bounded subset of R^p; and (5) $I(\theta) = \left[-E\left(\frac{\partial^2}{\partial\theta_i\partial\theta_j}\ln L\right)\right]$ $i, j = 1, 2, 3$ exists and is nonsingular. Thus, we can conclude that all regularity conditions are satisfied for the Burr type III distribution. Thus, an approximate $(1-\alpha)\%$ confidence interval on $\theta^T = (b, c, k)$ can be obtained by

$$\left(\hat{\theta}\_\theta\right) \underset{n\to\infty}{\overset{d}{\to}} N(0, I^{-1}(\theta)) \tag{A.1}$$

where $I(\theta) = \left[-E\left(\frac{\partial^2}{\partial\theta_i\partial\theta_j}\ln L\right)\right]$ $i, j = 1, 2, 3$. And the estimated equations for $\frac{\partial^2}{\partial\theta_i\partial\theta_j}\ln L\ \forall i, j = 1, 2, 3$ are given as follows:

$$\frac{\partial^2 \ln L}{\partial b^2} = \frac{n}{b^2} + \sum_{i=1}^{n}\left\langle\frac{-c-1}{b^2} - \frac{c\cdot(k+1)}{x_i^c}\cdot\left\{\frac{(c+1)\cdot b^{c-2}}{1+\left(\frac{b}{x_i}\right)^c} - b^{c-1}\cdot\left[1+\left(\frac{b}{x_i}\right)^c\right]^{-2}\cdot\left(\frac{c\cdot b^{c-1}}{x_i^c}\right)\right\}\right\rangle$$

$$\begin{aligned}\frac{\partial^2 \ln L}{\partial c^2} = \frac{-n}{c^2} + \sum_{i=1}^{n}\Bigg\langle -(k+1)\cdot\ln\left(\frac{b}{x_i}\right)\cdot\Bigg\{&\left(\frac{b}{x_i}\right)^c\cdot\ln\left(\frac{b}{x_i}\right)\cdot\left[1+\left(\frac{b}{x_i}\right)^c\right]^{-1} \\ &- \left(\frac{b}{x_i}\right)^{2c}\cdot\ln\left(\frac{b}{x_i}\right)\cdot\left[1+\left(\frac{b}{x_i}\right)^c\right]^{-2}\Bigg\}\Bigg\rangle\end{aligned}$$

$$\frac{\partial^2 \ln L}{\partial k^2} = \frac{-n}{k^2}$$

$$\begin{aligned}\frac{\partial^2 \ln L}{\partial b\partial c} = \sum_{i=1}^{n}\Bigg\langle\frac{1}{b} - (k+1)\cdot\Bigg\{&\frac{1}{x_i^c}\cdot c\cdot b^{c-1}\cdot\ln\left(\frac{b}{x_i}\right)\cdot\left[1+\left(\frac{b}{x_i}\right)^c\right]^{-1} + \left(\frac{b}{x_i}\right)^c\cdot\frac{1}{b}\cdot\left[1+\left(\frac{b}{x_i}\right)^c\right]^{-1} \\ &- c\cdot\left(\frac{b}{x_i}\right)^{2c-1}\cdot\left(\frac{1}{x_i}\right)\cdot\ln\left(\frac{b}{x_i}\right)\cdot\left[1+\left(\frac{b}{x_i}\right)^c\right]^{-2}\Bigg\}\Bigg\rangle\end{aligned}$$

$$\frac{\partial^2 \ln L}{\partial b\partial k} = -\sum_{i=1}^{n}\left[\frac{c\cdot b^{c-1}\cdot x_i^{-c}}{1+\left(\frac{b}{x_i}\right)^c}\right]$$

$$\frac{\partial^2 \ln L}{\partial c\partial k} = -\sum_{i=1}^{n}\left[\frac{\left(\frac{b}{x_i}\right)^c\cdot\ln\left(\frac{b}{x_i}\right)}{1+\left(\frac{b}{x_i}\right)^c}\right].$$

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